# A 4/3-Approximation Algorithm for the Minimum 2-Edge Connected Multisubgraph Problem in the Half-Integral Case

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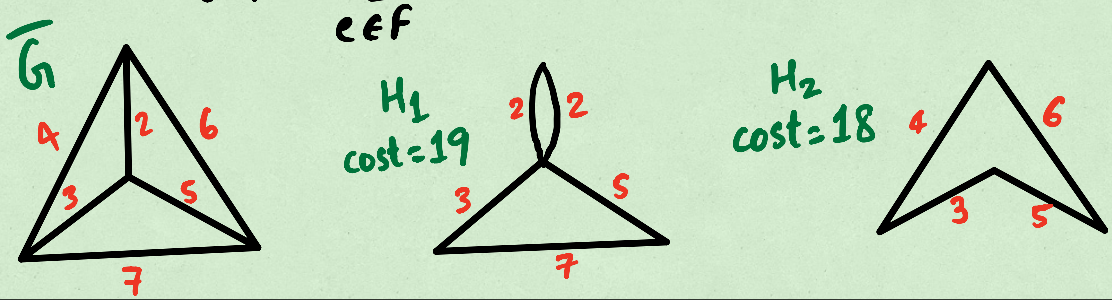
## 2-Edge Connected Multisubgraph Problem (2ECM)

Given: Undirected complete graph $\bar{G}=(\bar{V},\bar{E})$ on $n$ vertices

Metric costs $c:\bar{E}\to\mathbb{R}_+$

Goal: Find a 2-e.c. spanning multisubgraph $H=(V,F)$ of minimum cost

$c(H):=\sum_{e\in F} c_e$



## Some Remarks

- 2ECM is NP-hard
- Traveling Salesman Problem (TSP) = 2ECM + Eulerian Property

  even degree at vertices
- Strict version 2ECS:
  Given $G' = (V', E')$, nonnegative costs $c : E' \to \mathbb{R}_+$
  Find a 2-edge connected spanning subgraph of minimum cost

  usually techniques are diff & results incomparable

## Integer Linear Program for 2ECM

$$\min \sum_{e \in \bar{E}} c_e x_e$$

~~(2ECM-IP)~~ s.t.

(2ECM-LP) $\cong$ (Subtour-LP)

$$x(\delta(S)) \geq 2 \quad \forall\, \emptyset \subsetneq S \subsetneq \bar{V}$$

Subtour / 2-e.c. const.

$$x_e \geq 0 \quad \forall\, e \in \bar{E}$$

# of copies of e to include in our soln

$$\cancel{x_e \text{ integral}}$$

$$x(\delta(v)) = 2 \quad \forall\, v \in \bar{V}$$

- Parsimonious property [Goemans & Bertsimas '93]

## Subtour/Fractional 2ECM Polytope

- $$P := \left\{ x \in \mathbb{R}_+^{\bar{E}} : \begin{array}{l} x(\delta(v)) = 2 \quad \forall v \in \bar{V}, \\ x(\delta(S)) \geq 2 \quad \forall \emptyset \subsetneq S \subsetneq \bar{V} \end{array} \right\}$$

- $\text{LP-OPT} := \min\{ c^T x : x \in P \}$

- $\text{LP-OPT} \leq \text{OPT}_{2ECM} \leq \text{OPT}_{TSP} \leq \frac{3}{2} \cdot \text{LP-OPT}$

  Integrality gap $\alpha_{2ECM} \leq \alpha_{TSP} \leq 3/2$ ↑ Wolsey '80

- Recently, Karlin, Klein, and Oveis Gharan announced a $\left(\frac{3}{2} - 10^{-36}\right)$-approximation for TSP w.r.t. integer opt.

## Half-Integral Instances

An instance $(\bar{G}, c)$ for which (2ECM-LP)/(Subtour-LP) is optimized by $x$ s.t. $2x_e$ is integral $\forall e \in \bar{E}$

**Conjecture** [Schalekamp, Williamson, & van Zuylen '14]
Integrality gap of (Subtour-LP) for TSP is attained on half-integral instances

## What's known for such instances?

- [Carr & Ravi '98] LP-OPT $\leq$ $OPT_{2ECM}$ $\leq \frac{4}{3}$ LP-OPT

  constructive but not polytime

  $\alpha^{HI}_{2EM} \leq 4/3$

- [KKO '20] LP-OPT $\leq$ $OPT_{TSP} \leq (\frac{3}{2} - 0.00007)$ LP-OPT

  randomized apx. algo.

  $\frac{4}{3} \leq \alpha^{HI}_{TSP} < 3/2$

## Our Main Result

**Theorem 1** [Boyd et al. '20]

Let $x$ be an optimal half-integral solution to an instance $(\overline{G}, c)$ of 2ECM. In $O(n^2)$-time, we can compute a 2-e.c. spanning multisubgraph of $\overline{G}$ with cost at most $4/3\ c^T x$.

## Why consider such instances?

**Proposition** [Carr & Vempala '04]

Integrality gap of (2ECM-LP) $\leq \alpha$

$\Updownarrow$

For any integer $k \geq 2$, any $2k$-regular, $2k$-e.c. multigraph $G=(V,E)$, the uniform vector $\frac{\alpha}{k} \cdot \chi^E$ dominates a convex combination of incidence vectors of 2-e.c. spanning multisubgraphs of $G$.

coordinate-wise $\geq$

## Graph induced by $\frac{1}{2}$-integral $x \in \mathcal{P}$

- Define $G=(V,E)$ where $V := \overline{V}$ and $E$ has exactly $2x_e$ copies of edge $e \in \overline{E}$
- $x(\delta(v)) = 2 \;\; \forall v \in \overline{V} \Rightarrow G$ is 4-regular
- $x(\delta(S)) \geq 2 \;\; \forall \emptyset \subsetneq S \subsetneq \overline{V} \Rightarrow G$ is 4-e.c.

**Thm 2** [Carr & Ravi] Let $G=(V,E)$ be 4-reg, 4-e.c. multigraph and $e \in E$. There exists a finite collection $\{H_1, \dots, H_n\}$ of 2-e.c. spanning subgraphs of $G-e$ s.t. for some $\mu_1, \dots, \mu_n \geq 0$, $\sum_i \mu_i = 1$, we have:

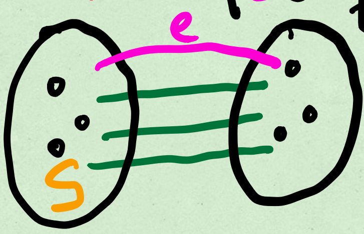
$$\frac{2}{3} \chi^{E \setminus \{e\}} = \sum_{i=1}^{n} \mu_i \chi^{H_i}$$

$\chi^{E(H_i)}$

# Proof Strategy of Carr & Ravi

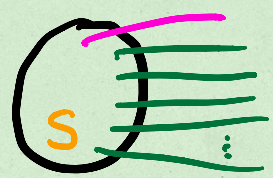
- Given 4-reg., 4-e.c. multigraph $G=(V,E)$
- Arbitrary edge $e \in E$
- Goal: Express $\frac{2}{3}\chi^{E-e}$ as a convex comb. of 2-e.c. subgraphs of $G$
- They give an inductive proof w/ two cases:

Case 1 ($e \in$ nontrivial tight cut $\delta(S)$)



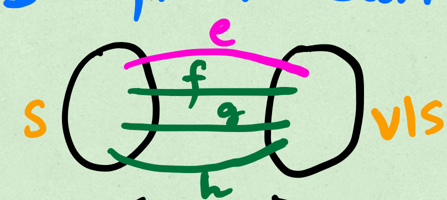

Case 2 (not case 1)

for any nontrivial $\delta(S) \ni e$



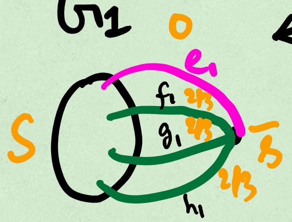

# Case 1 from Carr & Ravi

$G$

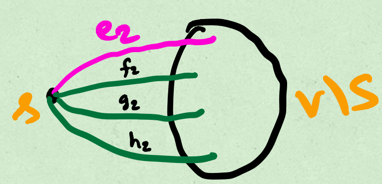



$|\delta(S)| = 4$

$G_1$ ← contract $V\setminus S$




contract $S$ → $G_2$




Glue

Convex comb. for $\frac{2}{3}\chi^{E(G_1)-e_1}$

mass in $\delta(\bar{s}) = 2$

Convex comb. for $\frac{2}{3}\chi^{E(G_2)-e_2}$

mass in $\delta(s) = 2$

Convex comb. for $\frac{2}{3}\chi^{E-e}$

## Case 2 from Carr & Ravi

$G$ has no non-trivial tight cuts $\ni e$

$G$

Apply two distinct splitting-off operations

n-1 vertices

$G_1 = (V_1, E_1)$ | $G_2 = (V_2, E_2)$

convex comb $\frac{2}{3}\chi^{E_1 - e_1}$ | convex comb $\frac{2}{3}\chi^{E_2 - e_2}$

lift comb. to $G$ | lift comb. to $G$

simple average

$\boxed{\text{Convex comb. } \frac{2}{3}\chi^{E-e}}$

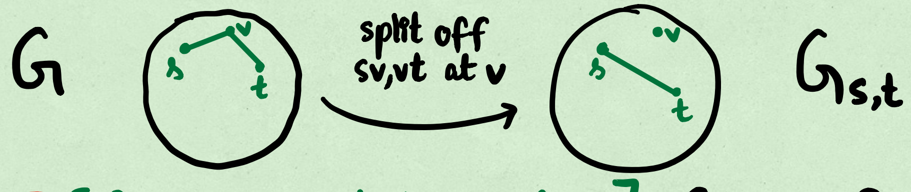
## Our simplifications of Carr & Ravi's Proof

- Get rid of the gluing step (unify analysis)
- Handle all cases by an extension of Lovász's splitting-off theorem, due to Bang-Jensen, Gabow, Jordán, and Szigeti.

**Theorem 3** [Boyd et al.] Let $G=(V,E)$ be 4-reg., 4-e.c. multigraph and $e \in E$. Let $c: E \to \mathbb{R}$ be arbitrary. Then, in $O(|V|^2)$-time, we can find 2-e.c. spanning subgraph $H$ of $G-e$ satisfying $c(H) \le \frac{2}{3} c(G-e)$.

## Preliminaries: splitting-off operation

**Def^n 4 [Splitting off]** Given multigraph $G$, two edges $sv$ and $vt$, the graph $G_{s,t}$ obtained by splitting off $(sv, vt)$ at $v$ is $G + st - sv - vt$.




**Def^n 5 [Complete splitting at v]** Given $G$ and vertex $v$ w/ even degree, a complete splitting at $v$ is a sequence of $\deg(v)/2$ splitting off operations (at $v$).

at the end $\deg(v) = 0$
delete $v$

## Prelims: Admissible pair

- $\lambda_H(x,y)$ := size of a minimum $(x,y)$-cut in $H$.

**Def^n 6 [Admissible pair]**

Let $k \geq 2$ be an integer. Let $G = (V+v, E)$ be a multigraph s.t. $\forall\ x, y \in V$, $\lambda_G(x,y) \geq k$.
Let $e = sv$ and $vt$ be two edges incident to $v$.
The pair $(sv, vt)$ is admissible if
$\forall\ x, y \in V$, $\lambda_{G_{s,t}}(x,y) \geq k$.

For any $e \in \delta(v)$, $A_e := \{ f \in \delta(v) \setminus \{e\} : (e,f) \text{ is admissible} \}$

## Extension of Lovász's splitting-off theorem

**Lemma 7** [Bang-Jensen et al. '99]
Let $k$ be even. Let $G=(V+v,E)$ s.t. $\forall\ x,y \in V$, $\lambda_G(x,y) \geq k$. Let $\deg_G(v)$ be even.
Then, $|A_{uv}| \geq \deg_G(v)/2$ for any $uv \in \delta(v)$.

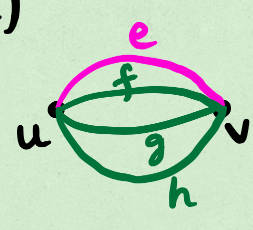
**Lemma 8:** Let $G$ be 4-reg., 4-e.c. and $e = vx \in \delta(v)$.
Then, (i) $|A_e| \geq 2$ ;
(ii) if $(e,f)$ is admissible for some $f = vy \in \delta(v)$, then the remaining two edges in $\delta(v) \setminus \{e,f\}$ are admissible.

## Simpler proof of Carr & Ravi's result

**Goal:** Show that $\forall$ 4-reg, 4-e.c. $G=(V,E)$, and $e \in E$, $\exists$ 2.e.c. subgraphs $H_1, \dots, H_r$, and convex co-eff $\mu_1, \dots, \mu_r$ s.t. $\frac{2}{3}\chi^{E-e} = \sum_i \mu_i \chi^{H_i}$

**Proof:** (By induction)

**Base case $n=2$ :**

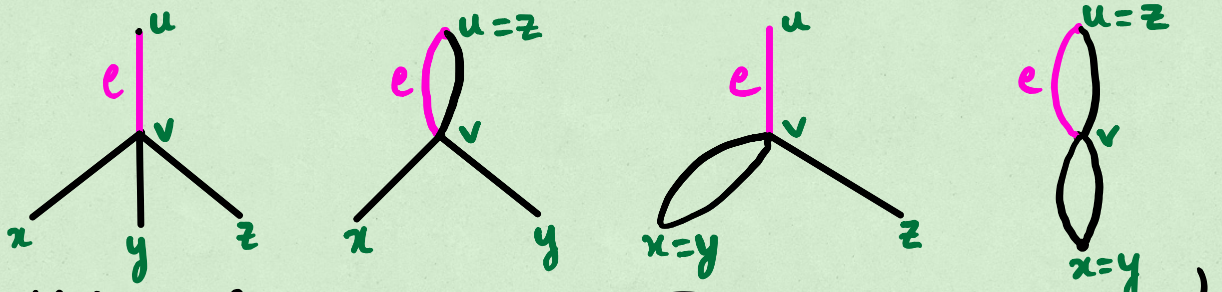



Observe $\frac{2}{3}\chi^{E-e} = \frac{1}{3}\left\{\chi^{\{f,g\}} + \chi^{\{f,h\}} + \chi^{\{g,h\}}\right\}$

# Inductive step

**General case:** $G$ is 4-reg, 4-e.c. and $e \in E$ designated edge
$(n \geq 3)$

four subcases




$|A_e| = 2$ or $3$

WLOG, assume $vx, vy \in A_e$

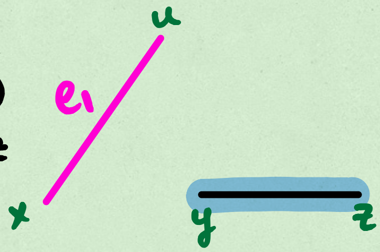
$A_e = \{vx, vy\}$

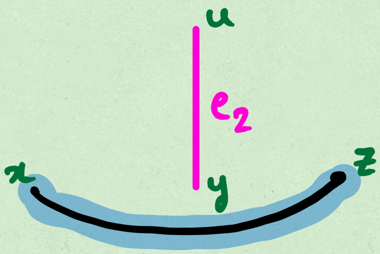
Two distinct complete splittings at $v$

$(e=uv, vx)$ then $(yv, vz)$ OR $(e=uv, vy)$ then $(xv, vz)$

# Two branches

| Split off $(e=uv, vx)$ & $(yv, vz)$ | Split off $(e=uv, vy)$ & $(xv, vz)$ |
| --- | --- |
| $V_1 = V - v$<br>$E_1 = E \setminus S(v) + ux + yz$<br>$e_1 = ux$ | $V_2 = V - v$<br>$E_2 = E \setminus S(v) + uy + xz$<br>$e_2 = uy$ |



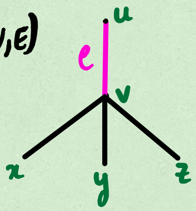




Apply induction on $G_i$ w/ designated edge $e_i$

| (Convex Comb $- G_1$) | (Convex Comb $- G_2$) |
| --- | --- |
| $\frac{2}{3}\chi^{E_1 - e_1} = \frac{2}{3}\chi^{E - S(v) + yz} = \sum_i \mu_i^1 \chi^{H_i^1}$ | $\frac{2}{3}\chi^{E_2 - e_2} = \frac{2}{3}\chi^{E - S(v) + xz} = \sum_i \mu_i^2 \chi^{H_i^2}$ |

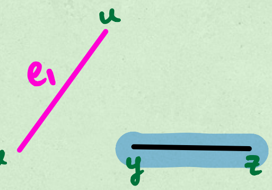
# Lift Operation

$G=(V,E)$




$\xrightarrow[\text{then } (yv,\, vz)]{\text{Split } (uv,\, vx)}$

$V_1 = V - v$
$E_1 = E \setminus \delta(v) + ux + yz$
$e_1 = ux$




(Convex Comb - $G_1$)

$$\frac{2}{3}\chi^{E-\delta(v)+yz} = \sum_i \mu_i^1 \chi^{H_i^1}$$

**Lift** each $H_i^1$ to a 2-e.c. subgraph of $G$:

$$\hat{H}_i^1 := \begin{cases} H_i^1 - yz + vy^{\,2/3} + vz^{\,2/3} & \text{if } yz \in E(H_i^1) \\ H_i^1 + vy^{\,1/3} + vx^{\,1/3} & \text{o.w.} \end{cases}$$

1/3

2/3 of the times

So, $$\sum_i \mu_i^1 \chi^{\hat{H}_i^1} = \frac{2}{3}\chi^{E-e} + \frac{1}{3}\left\{\chi^{vy} - \chi^{vx}\right\}$$

# Finishing the proof of Theorem 2

First branch gives $$\sum_i \mu_i^1 \chi^{\hat{H}_i^1} = \frac{2}{3}\chi^{E-e} + \frac{1}{3}\left\{\chi^{vy} - \chi^{vx}\right\}$$

By symmetry, second branch gives $$\sum_i \mu_i^2 \chi^{\hat{H}_i^2} = \frac{2}{3}\chi^{E-e} + \frac{1}{3}\left\{\chi^{vx} - \chi^{vy}\right\}$$

Averaging the above two combinations:

$$\frac{1}{2}\cdot\sum_i \mu_i^1 \chi^{\hat{H}_i^1} + \frac{1}{2}\cdot\sum_i \mu_i^2 \chi^{\hat{H}_i^2} = \frac{2}{3}\chi^{E-e}$$

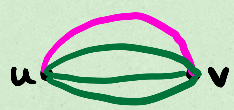
∎

# Algorithmic version: Theorem 3

Recall, Thm 3 [Boyd et al.] Let $G=(V,E)$ be 4-reg., 4-e.c. multigraph and $e \in E$. Let $c: E \to \mathbb{R}$ be arbitrary. Then, in $O(|V|^2)$-time, we can find 2-e.c. spanning subgraph $H$ of $G-e$ satisfying $c(H) \leq \frac{2}{3} c(G-e)$.

Proof: (Induction / Recursion)

Base case $n=2$




Choose the two cheapest green edges

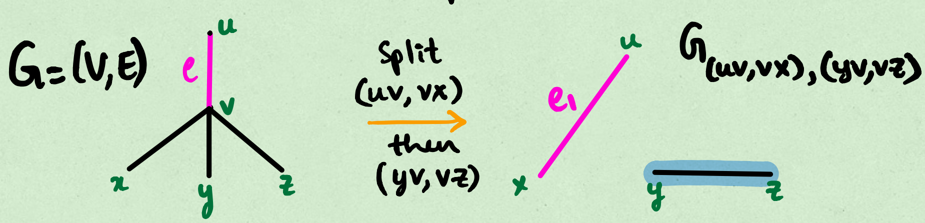
General case $n \geq 3$


- Recurse on the cheaper branch ← method of conditional expect.
- Compute $A_e$ in $O(n)$-time ← max-flow $|E| = 2n$
- WLOG, $vx \in A_e$ is the most expensive edge and $vy \in A_e \setminus \{vx\}$ be any other edge

# Proof of Theorem 3 (contd)

Choose $vx, vy \in A_e$ w/ $c_{vx} \geq c_{vy}$

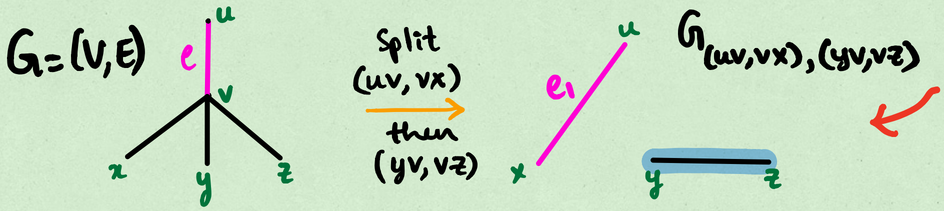
Then perform




- To recurse on the instance w/ $n-1$ vertices, we need to assign some cost to $yz$
- Let $\hat{H}$ be the recursive 2-e.c. subgraph satisfying $c(\hat{H}) \leq \frac{2}{3} c(G - v + ux + yz - ux)$

$$= \frac{2}{3} c(G-e) - \frac{2}{3}(c_{vx} + c_{vy} + c_{vz}) + \frac{2}{3} c_{yz}$$

# Finishing the proof of Theorem 3




$\hat{H}$ is a 2-e.c. subgraph w/ low cost

- **Hurdle:** The lift of $\hat{H}$ depends on $yz \overset{?}{\in} E(\hat{H})$

$$H := \begin{cases} \hat{H} - yz + vy + vz & \text{if } yz \in E(\hat{H}) \\ \hat{H} + vy + vx & \text{o.w.} \end{cases}$$

So, how can we control the cost in both cases?

- Define $c_{yz} := c_{vz} - c_{vx} \Rightarrow$ both cases increase in cost $= c_{vx} + c_{vy}$

($c_{yz} < 0$ is possible !!)

- Hence,

$$\begin{aligned} c(H) &= c(\hat{H}) + (c_{vx} + c_{vy}) \\ &\leq \frac{2}{3} c(G-e) - \frac{2}{3}(c_{vx} + c_{vy} + c_{vz}) + \frac{2}{3} c_{yz} \\ &\qquad + (c_{vx} + c_{vy}) \\ &= \frac{2}{3} c(G-e) + \frac{1}{3}(c_{vy} - c_{vx}) \leq \frac{2}{3} c(G-e) \quad \blacksquare \end{aligned}$$

by our choice, $c_{vx} \geq c_{vy}$

# $4/3$-approximation for 2ECM on half-integral instances

**Proof of Theorem 1:** (Given a half-integral instance $(\bar{G}, c)$ of 2ECM)

- Let $x$ be an optimal $1/2$-integral sol$^n$ to (2ECM-LP).
- Construct $G = (V, E)$ where $V := \bar{V}$ and

  $\forall e \in \bar{E}$, $E$ has $2x_e$ copies of $e$.

  Overload the same cost function onto $E$.
- Note: $G$ is 4-regular, 4-e.c.
- Apply Theorem 3 to $(G, c)$ w/ an arbitrary designated edge $e \in E$. We get a 2-e.c. spanning subgraph $H$ of $G$ s.t.

$$c(H) \leq 2/3\, c(G - e)$$

- Since $G$ is induced from $2x$,

$$c(G-e) \leq c(G) \leq 2c^T x$$

- $H$ is a $4/3$-approximate solution.

$\blacksquare$

# Conclusion

- Let $\alpha_{2ECM}$ denote the integrality gap of (2ECM-LP)
  $\alpha_{TSP}$ " ——— " (Subtour-LP)
- We saw a simpler proof of Carr & Ravi's result:

$$\alpha^{HI}_{2ECM} \leq 4/3$$

- We gave a matching approximation algo. for 2ECM on half-integral instances

Question 1: Efficient algo. for finding the convex combination?

Use Carr & Vempala's Meta Rounding algorithm

- Alexander, Boyd, and Elliot-Magwood showed $\alpha_{2ECM}$ on half-triangle pts $\geq 6/5$
- Boyd and Legault showed

$$\alpha^{HT}_{2ECM} \leq 6/5 \rightarrow$$ constructive polytime not known

Question 2: $$6/5 \leq \alpha^{HI}_{2ECM} \leq 4/3 \leq \alpha^{HI}_{TSP}$$

tight? strict? gap?

Question 3: Is $\alpha_{2ECM} < \alpha_{TSP}$ ?

Thank You !!

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